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Limit cycles and stochastic resonance in a periodically driven Langevin equation subject to white noise

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Abstract

We investigate stochastic resonance (SR) in a periodically driven Langevin system on a circle. The bifurcation parameter of the system and the amplitude of the periodic signal are controlled. The response amplitude that characterizes the degree of the amplification of the input periodic signal and the quality factor that is a coherence measure of the spontaneous periodic motion sustained by noise are computed and plotted as functions of the noise intensity. According to different parameter values, four types of behaviour are observed, as the results of interplay between two types of stochastic resonances, the spontaneous SR and conventional SR, respectively.

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1. Introduction

Since its introduction in 1980s by Benzi *et al* [1] and later by Nicolis *et al* [2], stochastic resonance (SR) has been the subject of current interest. For a recent review, see [3]. The term SR was originally referred to as a cooperative phenomenon in a nonlinear system, where a weak signal can be amplified and optimized by the assistance of noise [4–10]. Here three important ingredients, namely, a kind of threshold, a weak periodic modulation (signal) and a source of noise are thought to be necessary. Given these features, SR has been observed in a large variety of models, such as bistable ring laser [11], chemical reaction [12, 13], electronic system [14] and neuron systems [15–17], etc. Later, some authors found that SR can also happen in the absence of a periodic excitation [18–21]. This phenomenon of SR reflects a certain periodicity of the system driven only by white noise, here we introduce the name spontaneous SR to discriminate it from the conventional SR (i.e., with a periodic excitation).

Recently, the effective role of noise in the following Josephson junction system has attracted much attention [22, 23]

$$\alpha\ddot{\theta} + \beta\dot{\theta} + \sin\theta = b + A\cos\omega t + D\xi(t) \tag{1}$$

whereby θ is the phase difference across the junction, $\alpha = \hbar C/2e$, $\beta = \hbar/2eR$, b and A are the amplitudes of the dc and ac microwave components of the current through the junction, and $\xi(t)$ is the Gaussian white noise satisfying: $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(t')\rangle = \delta(t - t')$.

System (1) also describes the motion of a damped pendulum, with mass α, damping coefficient β, driven by torque and a periodic force plus white noise.

For system (1) without periodic driving ($A = 0$), the phenomenon of SR has been systematically investigated in our previous paper [26]. However, in the presence of a periodic driving, even the deterministic dynamics of the system is much more complicated. It may exhibit phase locking, chaos and hysteresis if the parameter is varied [25]. Nevertheless, when $\beta > 2\alpha$, the dynamics of system (1) in the deterministic case is clear, it has a one-dimensional periodic horizontal curve which attracts orbits exponentially [27]. Therefore the long-time behaviour of system (1) is similar to that of the following Langevin equation:

$$\dot{x} = b - \sin x + A\cos\omega t + D\xi(t). \tag{2}$$

Due to the close resemblance of the dynamics between system (2) and system (1) in the regime $\beta > 2\alpha$, in this paper, we give a systematic investigation of the interplay of noise and a periodic excitation in system (2). We will show new SR effects by controlling the bifurcation parameter b of the system and the amplitude A of the periodic signal. Precisely, for $b < 1$ and a weak periodic driving $A \ll 1$, besides a conventional SR as a best cooperative result of a periodic driving and noise, the system also preserves the original spontaneous SR sustained by noise; while for a subthreshold ($A < 1 - b$) but relatively large periodic driving, no conventional SR but only spontaneous SR exists. As for the case $b < 1$, $A > 1 - b$, the noise-background spectrum is well suppressed and only the conventional SR phenomenon exists.

The paper is organized as follows. In section 2, we will present a qualitative investigation of the phase portrait of the deterministic system (2) for $b < 1$ and some mathematical proofs, which makes the dynamical picture completely clear. Then in section 3, we will investigate the occurrence of SR as well as its physical interpretation based on the phase portrait.

2. Qualitative analysis of the deterministic dynamical behaviour

The deterministic dynamics of equation (2) can be characterized as

$$\dot{x} = b - \sin x + A\cos\omega t. \tag{3}$$

In the literature, the traditional way to explore the existence of a periodic solution is to take the signal term as a small perturbation. However, for large values of A, perturbation theory is no longer valid. To explore the dynamical behaviour of system (3) for both small and large periodic modulations, here we transform equation (3) into an autonomous system by setting $y = A\cos\omega t$ and $z = A\sin\omega t$, then equation (3) equivalently becomes

$$\begin{cases} \dot{x} = b - \sin x + y \\ \dot{y} = -\omega z \\ \dot{z} = \omega y. \end{cases} \tag{4}$$

Because of the periodicity in x, a solution to equation (4) can either be regarded as a curve winding on a cylinder $E^2 : y^2 + z^2 = A^2$ or on a torus $T = S^1 \times S^1$.

Let $\dot{x} = 0$, one has $b - \sin x + y = 0$, i.e. $y = \sin x - b$. Considering the intersection of the surface $y = \sin x - b$ with E^2 for $b < 1$, there are two different cases:

Case 1. $A \leqslant 1 - b$, where the phase $y = \sin x - b$ divides the cylinder into three parts in every strip $[2k\pi - \pi/2, 2k\pi + 3\pi/2]$ of x. In every neighbouring part, $\dot{x}$ changes sign (see figure 1(a)).

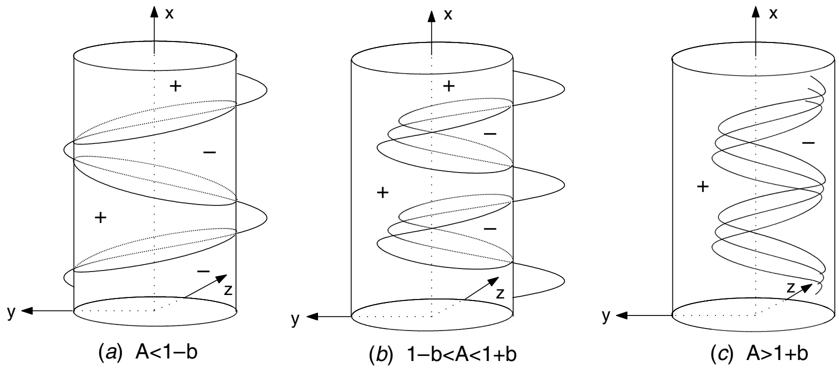

Figure 1. The intersecting cases of the surface $y = \sin x - b$ with cylinder E^2. (a) $0 < A < 1-b$, (b) $1-b < A < 1+b$ and (c) $A > 1+b$, where the sign '+' represents $\dot{x} > 0$, while '−' represents $\dot{x} < 0$.

Case 2. $A > 1-b$, where the phase $y = \sin x - b$ divides the cylinder into two sections in every strip $[2k\pi - \pi/2, 2k\pi + 3\pi/2]$ of x. In this case, the intersection for $1-b < A < 1+b$ is like figure 1(b) and that for $A > 1+b$ is like figure 1(c).

In the following we will investigate the dynamical behaviour of system (4) for $b < 1$ in these two cases, respectively. The case $b > 1$ is of no interest [20].

2.1. Case 1: $A \leqslant 1 - b$

For the periodicity of $\sin x$, we only need to consider $x \in (-\pi/2, 3\pi/2)$.

Let

$$G_1 : \{(t, x) | t \in R, -\pi/2 < x \leqslant \pi/2\} \qquad G_2 : \{(t, x) | t \in R, \pi/2 < x \leqslant 3\pi/2\}$$

be two ring-shape regions on the cylinder E^2. Obviously, there is no equilibrium point in G_1 and G_2. Any orbit moving along the boundary of G_1 will eventually enter G_1 and any orbit moving along the boundary of G_2 will eventually leave G_2. So by Poincaré–Bendixion theorem on a cylinder, at least one stable limit cycle (SLC) lies in region G_1 and at least one unstable limit cycle (ULC) lies in G_2.

In the following, we will prove that there exists at most one SLC in G_1 and at most one ULC in G_2. Otherwise, if there are two limit cycles in G_1, then the corresponding solutions to (4) can be written as

$$\mathrm{LC}_1 : \begin{cases} x_1 = f_1(t) \\ y_1 = A\cos\omega t \\ z_1 = A\sin\omega t \end{cases} \qquad \mathrm{LC}_2 : \begin{cases} x_2 = f_2(t) \\ y_2 = A\cos\omega t \\ z_2 = A\sin\omega t. \end{cases}$$

Obviously, LC_1 and LC_2 should have the same periodicity $T = \frac{2\pi}{\omega}$.

It follows from equation (4) that

$$\dot{x}_2(t) - \dot{x}_1(t) = \sin x_1 - \sin x_2.$$

Then

$$[x_2(t) - x_1(t)] - [x_2(0) - x_1(0)] = \int_0^t (\sin x_1 - \sin x_2)\,\mathrm{d}t.$$

Suppose $x_1 < x_2$. Since $x_1, x_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $\sin x_1 - \sin x_2 < 0$. As a result, $[x_2(T) - x_1(T)] - [x_2(0) - x_1(0)] < 0$. This contradicts the fact that $x_1(t), x_2(t)$ are two periodic solutions to equation (4). Therefore, there is only one SLC in region G_1. Similarly, only one ULC can exist in G_2.

2.2. Case 2: $A > 1 - b$

Firstly, let us investigate the situation for $\omega \gg 1$. We introduce the following transformation:

$$\Gamma : \begin{cases} \xi = \mathrm{e}^x y \\ \eta = \mathrm{e}^x z. \end{cases}$$

Obviously, the transformation Γ is a homeomorphism. It maps a ring-shaped region on E^2 to a ring-shaped region on the ξ–η plane and maps a cycle $C : x = x_c$ on E^2 to a circle $C' : \xi^2 + \eta^2 = A^2\,\mathrm{e}^{2x_c}$ on the ξ–η plane. The inverse transformation of Γ can be expressed as

$$\Gamma^{-1} : \begin{cases} x = \frac{1}{2}\ln\left(\frac{\xi^2}{A^2} + \frac{\eta^2}{A^2}\right) \\ y = \xi\left(\frac{\xi^2}{A^2} + \frac{\eta^2}{A^2}\right)^{-\frac{1}{2}} \\ z = \eta\left(\frac{\xi^2}{A^2} + \frac{\eta^2}{A^2}\right)^{-\frac{1}{2}}. \end{cases}$$

So we only need to study the dynamics of the following equivalent system:

$$\begin{cases} \frac{\mathrm{d}\xi}{\mathrm{d}t} = \dot{x}\xi - \omega\eta \\ \frac{\mathrm{d}\eta}{\mathrm{d}t} = \dot{x}\eta + \omega\xi \end{cases} \tag{5}$$

where $\dot{x} = b - \sin\left(\frac{1}{2}\ln\frac{\xi^2+\eta^2}{A^2}\right) + \xi\left(\frac{\xi^2}{A^2} + \frac{\eta^2}{A^2}\right)^{-1/2}$.

Let $t = \frac{s}{\omega}$, then

$$\begin{cases} \frac{\mathrm{d}\xi}{\mathrm{d}t} = -\eta + \frac{1}{\omega}\dot{x}\xi \\ \frac{\mathrm{d}\eta}{\mathrm{d}t} = \xi + \frac{1}{\omega}\dot{x}\eta. \end{cases} \tag{6}$$

To judge the existence of limit cycles of system (6) for $\omega \gg 1$, we cite the following lemma without proof ([28]):

Lemma 2.1. *Let*

$$\begin{cases} \dot{x} = -y + \lambda f_1(x, y) \\ \dot{y} = x + \lambda f_2(x, y) \end{cases} \tag{7}$$

be a perturbation of a linear dynamical system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x. \end{cases} \tag{8}$$

Suppose that the equilibrium point $(0, 0)$ *of equation (8) is still the unique equilibrium of system* (7)*, but no longer of central type for* $\lambda \neq 0$*. Let*

$$\Phi(r) = \int_0^{2\pi} [x f_1(x, y) + y f_2(x, y)]\,\mathrm{d}t$$

where $x = r\sin t$, $y = r\cos t$*, then*

(1) For $\lambda \ll 1$*, the necessary condition for equation (7) to have a closed orbit near the orbit* $L_{r_0} : x = r_0 \sin t$, $y = r_0 \cos t$ *of equation (8) is* $\Phi(r_0) = 0$.

(2) If $r_0 > 0, \Phi(r_0) = 0$ *and* r_0 *is not the extreme point of* $\Phi(r_0)$*, then for* $\lambda \ll 1$*, equation (7) has a closed orbit near* L_{r_0}.

(3) If $\Phi(r_0) = \cdots = \Phi^{(2k)}(r_0) = 0$*, and* $\Phi^{(2k+1)}(r_0) < 0$*, then for* $\lambda \ll 1$*, equation (7) has a limit cycle near* L_{r_0}*. It is stable for* $\lambda > 0$ *and unstable for* $\lambda < 0$.

For equation (6), let $\xi = r\sin s$, $\eta = r\cos s$, $f_1 = \xi\dot{x}$, $f_2 = \eta\dot{x}$, we have

$$\Phi(r) = \frac{2\pi r}{\omega}\left(b - \sin\left(\ln\frac{r}{A}\right)\right).$$

The roots of $\Phi(r) = 0$ are: $r_0 = 0$ (discarded), $r_1 = A\exp(\arcsin b)$ and $r_2 = A\exp(\pi - \arcsin b)$, the corresponding first derivatives are: $\Phi'(r_1) = -\frac{2\pi}{\omega}\sqrt{1-b^2} < 0$ and $\Phi'(r_1) = \frac{2\pi}{\omega}\sqrt{1-b^2} > 0$.

According to lemma 2.1, we know that for $\omega \gg 1$, system (6) has a SLC at $\xi = r_1\sin s, \eta = r_1\cos s$ and a ULC at $\xi = r_2\sin s, \eta = r_2\cos s$. Except for these two limit cycles, no other limit cycle exists for $x \in (-3\pi/2, \pi/2]$. Correspondingly, system (4) has a unique SLC at $x = \arcsin b$ and a unique ULC at $x = \pi - \arcsin b$ in $(-3\pi/2, \pi/2]$.

Now let us consider the situation for $\omega \ll 1$. Here we take the case $1 - b < A < 1 + b$ for example to prove the nonexistence of the limit cycle. Let $H_k : \{(z, y, x) | 2k\pi - \pi/2 < x \leqslant 2k\pi + 3\pi/2\}$ be a ring-shaped region of system (4) on the cylinder ($k \in Z$). If there is a limit cycle, obviously it could not interact with both H_k and H_{k+1}. Otherwise, it contradicts the direction of the vector fields. So the limit cycle can only exist in the region H_k, hence its amplitude is smaller than 2π.

Let $\{(z(t), y(t), x(t))\}_{t\geqslant 0}$ be any limit cycle in H_k, $a = \frac{A-(1-b)}{2}$, and

$$F(t) = \sin\omega t - \frac{1-b+a}{A}\omega t.$$

Then $F(0) = 0$, $F'(t) = \omega\big(\cos\omega t - \frac{1-b+a}{A}\big)$. If $t \in \big(0, \frac{\arccos\frac{1-b+a}{A}}{\omega}\big]$, then $F'(t) \geqslant 0$. For sufficiently small ω $\big(0 < \omega < \frac{a}{2\pi}\arccos\frac{1-b+a}{A}\big)$, we can select $t_1 \in \big(\frac{2\pi}{a}, \frac{\arccos\frac{1-b+a}{A}}{\omega}\big)$, such that $t_1 a > 2\pi$. Then $F(t_1) \geqslant 0$, i.e.,

$$\frac{\sin\omega t_1}{\omega t_1} \geqslant \frac{1-b+a}{A}.$$

So for these ω, we have

$$\begin{aligned} x(t_1) - x(0) &= bt_1 - \int_0^{t_1}\sin x(s)\,\mathrm{d}s + A\frac{\sin\omega t_1}{\omega} \\ &> t_1\left(b - 1 + A\frac{\sin\omega t_1}{\omega t_1}\right) \\ &> t_1\left(b - 1 + A\frac{1-b+a}{A}\right) \\ &= t_1 a > 2\pi. \end{aligned}$$

This contradicts the fact that a periodic solution can only lie in the region H_k with the amplitude smaller than 2π. So for $\omega \ll 1$, and $1 - b < A < 1 + b$, system (3) has no limit cycle on the cylinder.

Remark. Summarizing the above two cases, we know that for every fixed value of b, there exists a critical function $\omega = \omega_c(A)$ such that for $\omega > \omega_c(A)$, system (4) has two limit cycles, while for $\omega < \omega_c(A)$, no limit cycle exists.

3. Numerical simulations for stochastic resonance

Based on the above mathematical analysis, now let us investigate the influence of noise on system (4). To measure noise-induced enhancement of the amplitude of a periodic signal, we average power spectra of 150 runs of the time series of $\{\sin x(t)\}_{t\geqslant 0}$ and calculate the

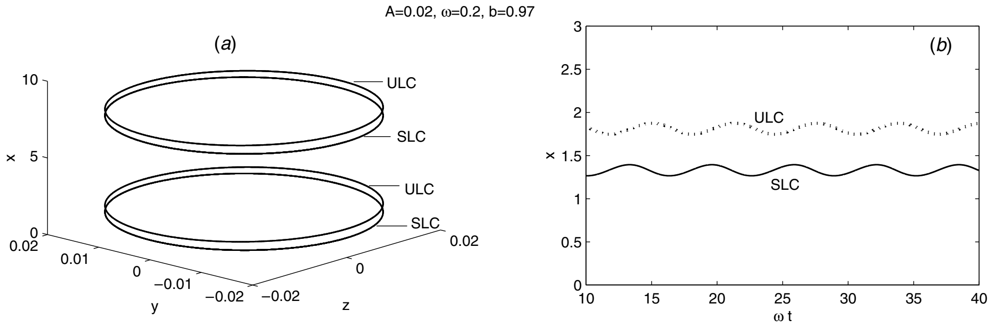

Figure 2. The phase portrait of system (4) for $b = 0.97$, $A = 0.02$, $\omega = 0.2$: (*a*) the stable limit cycles and unstable limit cycles on the cylinder E^2 for two neighbouring periodic strips of x, (*b*) the limit cycles cut along one generator of the cylinder and developed to a plane for one period strip of x.

response amplitude RA to characterize the occurrence of SR. Here RA $= \frac{R_1}{R_0}$, where R_1 is the amplitude of the output at the driving frequency in a noisy background and R_0 is the one in the deterministic case.

3.1. $A \leqslant 1 - b$

In this parameter region, for every driving frequency ω, the deterministic system (4) has a stable limit cycle and a unstable limit cycle on every strip $(2k\pi - \pi/2, 2k\pi + 3\pi/2]$ $(k = 0, \pm 1, \pm 2, \ldots)$. To investigate the behaviour of the system in the presence of noise perturbation, here we consider different sets of system parameters. Firstly, let us see the situation for a weak periodic driving (we take $b = 0.97$, $A = 0.02$, $\omega = 0.2$ for illustration). Figure 2 is the phase portrait of the deterministic system, where SLCs with $2k\pi$ phase difference are displayed. And figure 3(*a*) shows the power spectrum of the deterministic output. There one sees a short spectrum peak at the driving frequency. When noise is included, hopping motion from one SLC to the next SLC via a ULC on the cylinder happens. Reflected in the power spectrum, there is an enhanced spectrum peak at the driving frequency (see figure 3(*b*)). With the noise intensity D increasing, the height of this peak correspondingly increases until reaching a maximum at about $D = 0.3$, which manifests the best degree of coherence of the hopping motion. After that, the spectrum peak decreases with the further increase of D. This means that in system (2) the conventional SR happens as the best cooperative result of a weak periodic signal and white noise. To confirm this, we also plot the curve of RA versus D in figure 4(*a*).

In figure 3, besides a sharp peak at the driving frequency, one can also see a distinct noise-background spectrum with a profile very similar to the spectrum of the following Langevin equation,

$$\dot{x} = b - \sin x + D\xi(t) \tag{9}$$

i.e., system (2) without periodic driving case. So the appearance of a spectrum peak at another nonzero frequency can be regarded as reflecting the noise-induced coherence before the periodic driving is applied. Calculating the quality factor β of the background power spectrum shows that the spontaneous SR which occurs in system (1) is still preserved in the presence of periodic modulations (see figure 4(*b*)). Here β is taken as $\beta = \omega_p h/W$, where h is the height of the peak of the noise-background power spectrum (exclude the spectrum peak

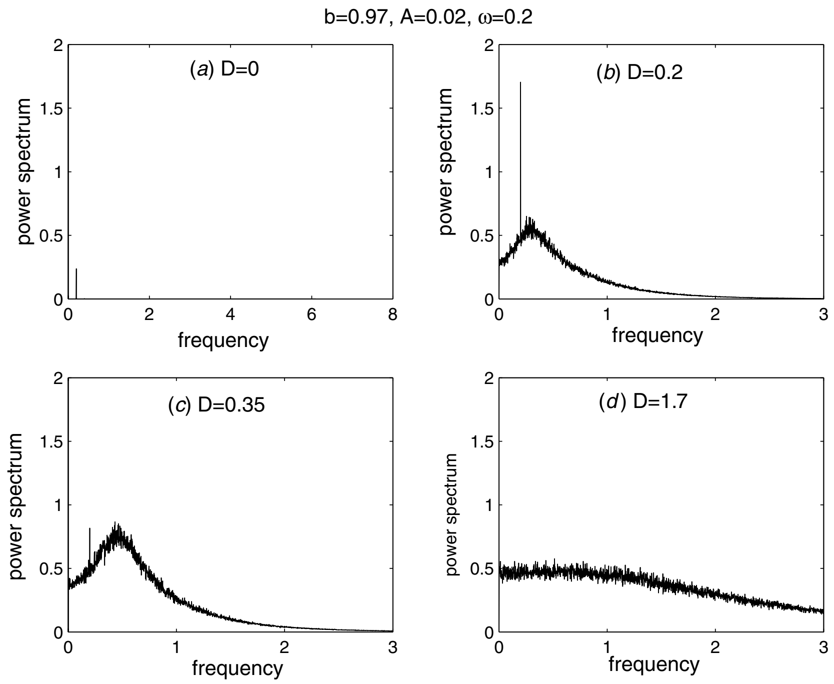

Figure 3. The power spectra for different noise intensities of system (2) for $b = 0.97$, $A = 0.02$, $\omega = 0.2$.

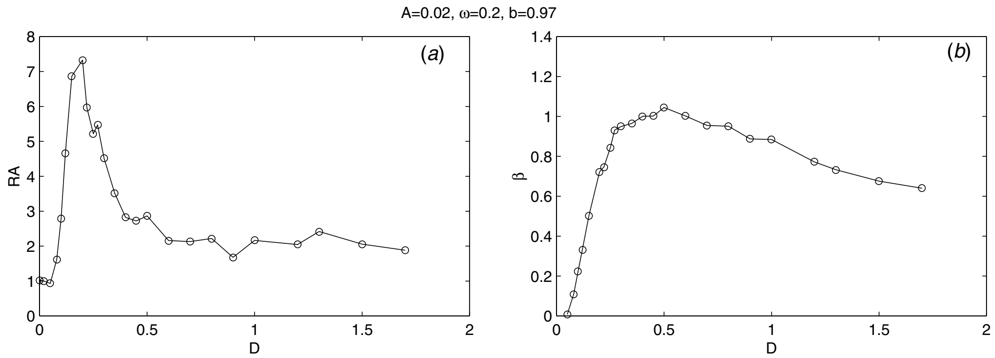

Figure 4. (*a*) The response amplitude RA versus the noise intensity D, (*b*) the quality factor β of the noise-background spectrum versus D of system (2) for $b = 0.97$, $A = 0.02$, $\omega = 0.2$.

at the driving frequency), W is the width of the spectrum measured at the height $h/\sqrt{e}$ and ω_p is the peak frequency [24].

Now let us explore the situation when the amplitude of the driving signal is relatively larger. For illustration, here we take $b = 0.9$, $A = 0.09$ and $\omega = 0.4$. Figure 5 is the deterministic phase portrait, where limit cycles (or periodic solutions) are displayed. And in figure 6, power spectra for different noise intensities are plotted. One can see that when noise is introduced, the power spectrum is still composed of two distinct parts: a sharp peak at the input frequency and the noise-background spectrum with the peak at another

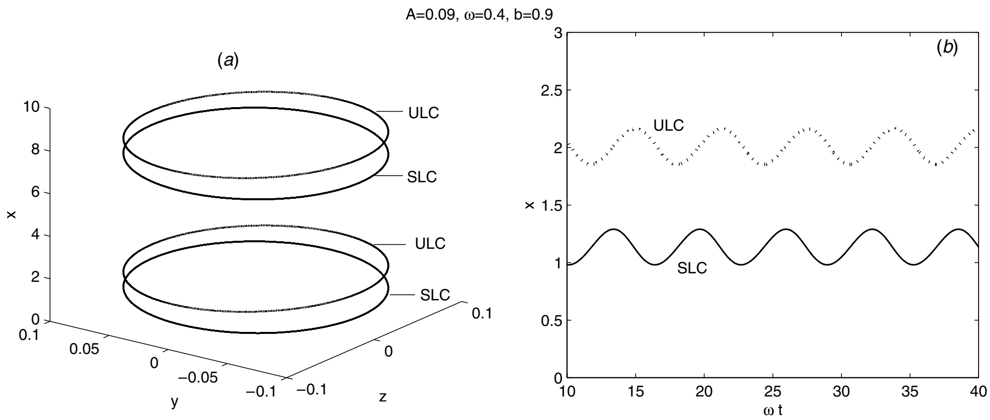

Figure 5. The phase portrait of system (4) for $b = 0.9$, $A = 0.09$, $\omega = 0.4$: (*a*) the stable limit cycles and unstable limit cycles for two neighbouring periodic strips of x, (*b*) the limit cycles cut along one generator of the cylinder and developed to a plane for one period strip of x.

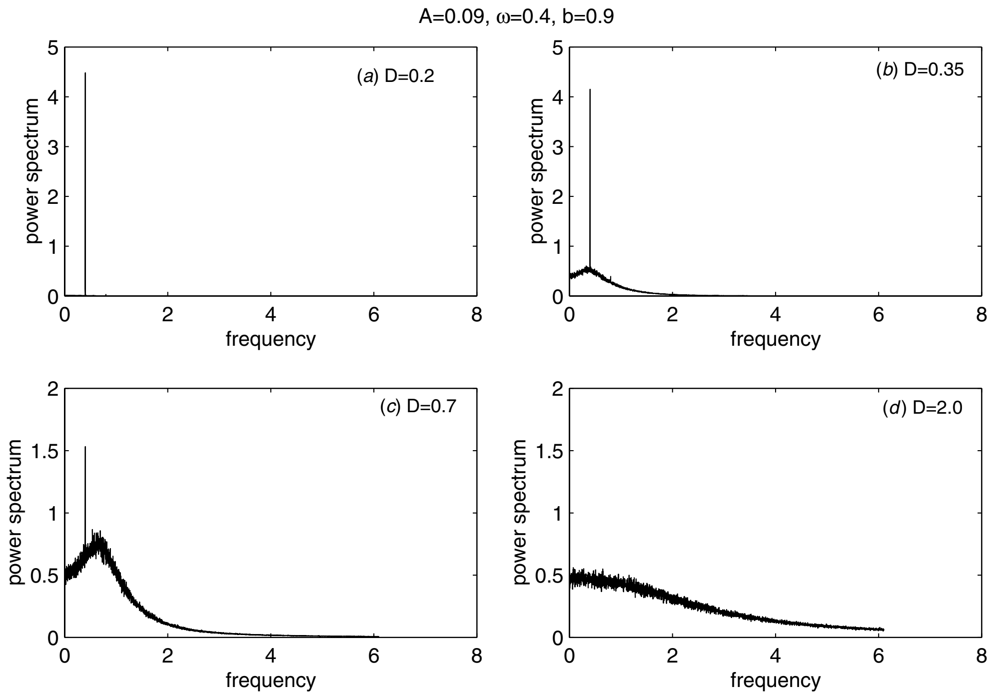

Figure 6. The power spectra of system (2) for $b = 0.9$, $A = 0.09$, $\omega = 0.4$.

frequency. Nevertheless, with the increase in the noise strength, the height of the spectrum peak at the driving frequency is decreasing. So for this set of parameters, no conventional SR happens (see figure 7(*a*)). However, the bell-shaped curve of the quality factor β versus D in figure 7(*b*) shows that there still exists spontaneous SR happening at $D \approx 0.7$.

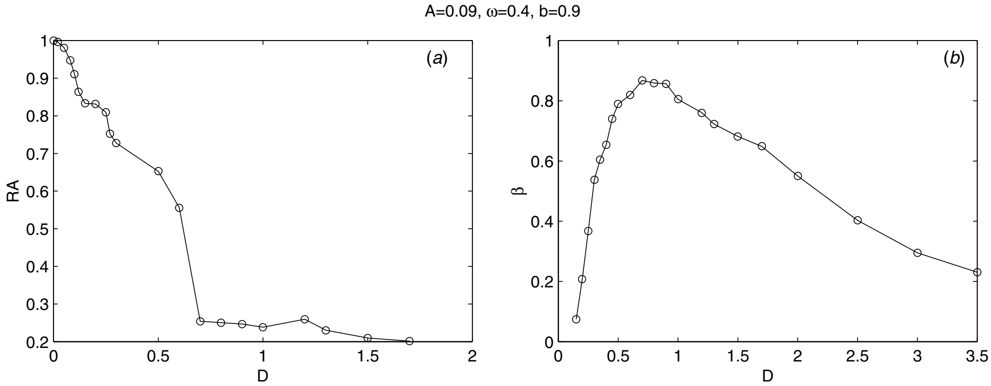

Figure 7. (*a*) The response amplitude RA versus the noise intensity D, (*b*) the quality factor β of the noise-background spectrum versus D of system (2) for $b = 0.9$, $A = 0.09$, $\omega = 0.4$.

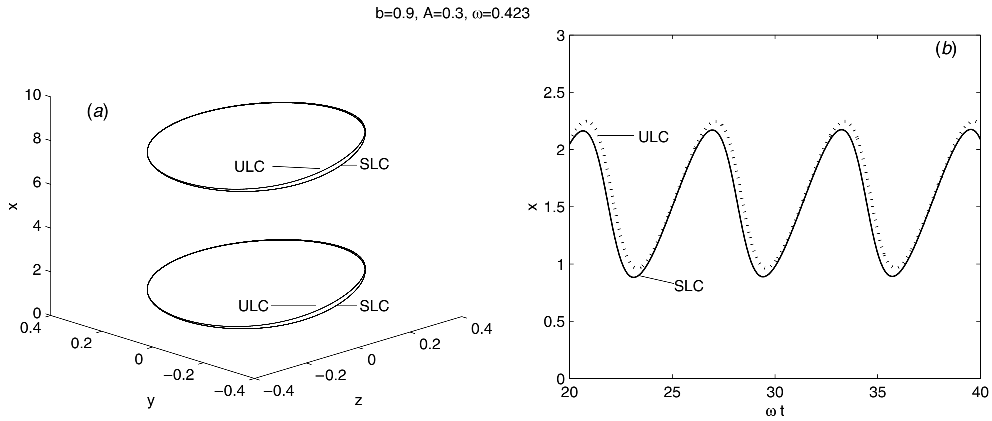

Figure 8. The phase portrait of system (4) for $b = 0.9$, $A = 0.3$, $\omega = 0.423$: (*a*) the stable limit cycles and unstable limit cycles for two neighbouring periodic strips of x, (*b*) the limit cycles cut along one generator of the cylinder and developed to a plane for one period strip of x.

3.2. $A > 1 - b$

Furthermore, let us investigate the case for superthreshold periodic modulations ($A > 1 - b$). For not too large amplitude of the periodic modulation, system (4) has limit cycles on the cylinder. Then what will the phenomenon be in this superthreshold case? Here we take the parameters $b = 0.9$, $A = 0.3$, $\omega = 0.423$ for illustration. After plotting the phase portraits in figure 8, we plot the power spectra for different noise intensities in figure 9. We find that the noise-background spectrum is now effectively suppressed. So in this case, no spontaneous SR occurs. However, with the increase in the noise strength, the response amplitude (RA) versus the noise intensity D undergoes a resonance-like process. This just characterizes the occurrence of the conventional SR.

Combining the phenomena for different sets of system parameters in section A, we learn that for a weak periodic driving ($A \ll 1$), there occurs not only the conventional SR in system (2), but also the spontaneous SR which has occurred without periodic driving is preserved. However, for a subthreshold but relatively larger periodic modulation, the output at the driving

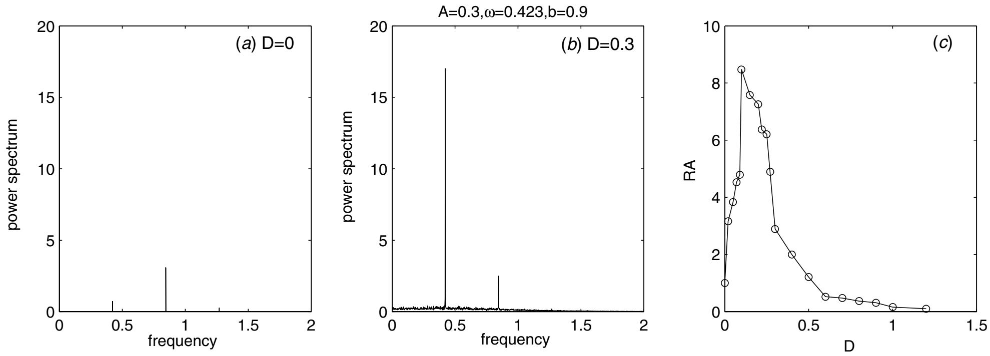

Figure 9. The power spectra for different noise intensities: (*a*) $D = 0$, (*b*) $D = 0.3$; (*c*) the response amplitude RA versus D for $b = 0.9$, $A = 0.3$, $\omega = 0.423$.

frequency is suppressed. Then there is no conventional SR phenomenon, but spontaneous SR can still be observed. Notwithstanding, when the strength of a periodic input exceeds $1 - b$, the noise-background spectrum is effectively suppressed and the output at the driving frequency is greatly enhanced. Consequently, no spontaneous SR, but only conventional SR happens as the best cooperative result of white noise and a periodic driving.

The reason for the above different responses of the system is not difficult to understand in the following way. Without periodic driving, it has been clarified in [20] that for every fixed noise intensity (not too large), the power spectrum of the output of system (9) has a wide range of frequencies with finite spectrum height. When a periodic driving is applied: (i) for $A \ll 1$, since the periodic driving is weak, then it can hardly concentrate the signals distributed widely on different frequencies (see the obvious noise-background spectrum in figure 3(*c*)) just close to the driving frequency. So besides keeping synchronization with the external periodic driving and the conventional SR occurring at an optimal noise intensity, system (2) still preserves the original noise-induced behaviour and there happens spontaneous SR at another suitable noise strength. (ii) If a relatively large but still subthreshold modulation is applied, one can see from figure 5 that the distance between a SLC and its neighbouring ULC is much larger than that in figure 2 which corresponds to the case $A \ll 1$. Then hopping between SLCs in this case becomes much more difficult than in the case $A \ll 1$. Since the system spends more time wandering near the SLC, and larger amplitude of the periodic modulation corresponds to larger value of R_0 which is not good for boosting the response amplitude, so the output decreases as compared to the case without noise perturbation. Hence no conventional SR occurs. But the periodic modulation in this case is still not strong enough to change the original noise-induced behaviour, so spontaneous SR is still preserved. (iii) For the superthreshold modulations ($A > 1 - b$), figure 8 shows that the ULC is very close to the SLC, then only a weak noise perturbation can induce the hopping motions between the SLCs with a good degree of coherence. So the noise-background spectrum is well suppressed and the output at the driving frequency is greatly facilitated. Thus only conventional SR occurs at an optimal noise intensity.

So far, we have manifested the different interplay results of noise and periodic driving in the cases when system (4) has limit cycles. Now let us further investigate the situation when the deterministic system (4) has no limit cycle on the cylinder. Figure 10 is the phase portrait of system (4) for $b = 0.9$, $A = 0.4$ and $\omega = 0.2$. It is shown that without any noise perturbation, solutions of the deterministic system, which wind around the cylinder E^2, are

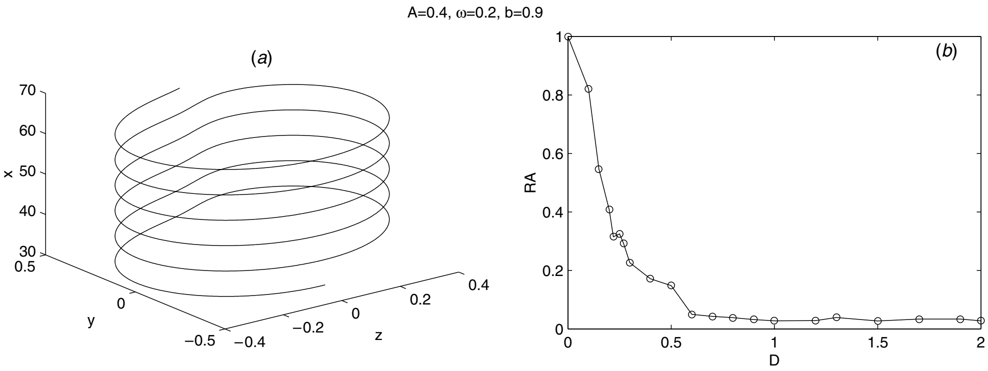

Figure 10. (*a*) The phase portrait of system (4), (*b*) the response amplitude RA versus D for $b = 0.9$, $A = 0.4$, $\omega = 0.2$.

all running periodic. Then it is easy to conclude that adding noise can only destroy such a regular motion. This is confirmed in figure 10(*b*), where the response amplitude of the output is decreasing with the increase in the noise level. Numerical simulations for other sets of parameters (for which the deterministic system has no limit cycle) also show the same result. Therefore it is impossible for SR to exist when system (4) has no limit cycle.

4. Concluding remarks

We have given a systematic investigation of SR in an over-damped Langevin system driven by a periodic signal plus white noise, based on qualitative analysis of the dynamics of the corresponding deterministic system. It has been shown that for a subthreshold periodic driving ($A < 1 - b$), there exist a unique stable periodic solution and a unique unstable periodic solution in every strip $x \in (2k\pi - \pi/2, 2k\pi + 3\pi/2]$ on the cylinder E^2; and for a superthreshold periodic modulation, there exists a critical value $\omega_c(A)$ such that for $\omega > \omega_c(A)$, the situation is the same as the subthreshold case, while for $\omega < \omega_c(A)$, no periodic solution exists.

Secondly, based on such clear dynamical pictures, SR are shown to exist only when the deterministic system has stable periodic solutions. And for different periodic modulations, the noise-driven system has different responses. More precisely, for a weak subthreshold periodic driving, both conventional SR and spontaneous SR are observed; while for a relatively large but still subthreshold driving, only spontaneous SR occurs; as for a superthreshold driving, no spontaneous SR but only conventional SR exists.

Thirdly, though the considered model is simple, the results we obtained here are generic. We will further show in our forthcoming work that for systems of N overdamped pendula coupled to their nearest neighbours

$$\dot{x}_i = b - \sin x_i + K(x_{i-1} + x_{i+1} - 2x_i) + A\cos\omega t + D_i\xi(t) \tag{10}$$

similar phenomena can be observed. Even more, the results can be further extended to the Josephson junction arrays governed by the damped, driven, discrete sine-Gordon equation

$$\ddot{x}_i + \alpha\dot{x}_i = b - \sin x_i + K(x_{i-1} + x_{i+1} - 2x_i) + A\cos\omega t + D_i\xi(t) \tag{11}$$

when the damping coefficient α is not very small. Precise discussions will be presented in our subsequent papers.

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